$$\Delta^2 A = -\frac{(dt)^2}{2} \int_D \rho \, (\Delta \mathrm{w})^2 \, d\tau < 0$$

Therefore, it has been established for a continuous medium [8, 9],

Chetaev principle. The work of an elementary cycle consisting of the forward motion of a continuous medium in the field of mass and surface forces and the reverse motion in a field of forces which would be sufficient to produce the actual motion if the medium particles were perfectly free, has (at least a relative) maximum in the class of fictitous Gaussian motions for the actual motion.

Just as the D'Alembert-Lagrange principle, this principle can also be expressed taking into account the first and second laws of thermodynamics.

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METHOD OF SUCCESSIVE APPROXIMATIONS FOR THREE-DIMENSIONAL LAMINAR BOUNDARY LAYER PROBLEMS (LOCALLY SELF-SIMILAR CASE)

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We present an analytical method for the computation of problems of incompressible boundary layer theory based on an application of the method of successive approximations. The system of equations is reduced to a form suitable for integration. Parameters characterizing the external flow and the body geometry are contained only in the coefficients of the system and do not enter into the boundary conditions. The transformed momentum equations are inte-

grated across the boundary layer from a current value to infinity with the boundary conditions taken into account. If the integration is made from zero to infinity, then the equations pass over into the Kármán relations. Integrating the system of equations a second time, using the boundary conditions at the wall, we obtain a system of nonlinear integro-differential equations. To solve this system of equations we apply the method of successive approximations. To satisfy the boundary conditions at infinity we introduce, at each step of the iterations, unknown "governing" functions. From the conditions at the outer side of the boundary layer we obtain additional equations for their determination. With the iterational algorithm formulated in this way, the boundary conditions, both on the body and at the outer side of the boundary layer; are satisfied automatically.

We consider a locally self-similar approximation. In this case, relative to the "governing" functions, we obtain an algebraic system of equations. We write out the solution in the first approximation. The results obtained in the first approximation are compared with the results of finite-difference computations for a wide range of problems. The results obtained in this paper are compared with those obtained in [1] for the flow in the neighborhood of a stagnation point. An indication is given of the nonuniqueness of the solutions of the three-dimensional boundary layer equations.

1. We consider the flow of a viscous incompressible fluid over an arbitrary smooth surface s. The three-dimensional boundary layer equations for an incompressible fluid, obtained under the usual assumptions of boundary layer theory, have the form [2]

$$\frac{u}{\sqrt{g_{11}}}\frac{\partial u}{\partial \xi} + \frac{\omega}{\sqrt{g_{22}}}\frac{\partial u}{\partial \eta} + v\frac{\partial u}{\partial \zeta} + A_1 u^2 + A_2\omega^2 + A_3 u\omega = A_4 + \nu\frac{\partial^2 u}{\partial \zeta^2} \qquad (1.1)$$

$$\frac{\partial p}{\partial \zeta} = 0$$

$$\frac{u}{\sqrt{g_{11}}}\frac{\partial \omega}{\partial \xi} + \frac{\omega}{\sqrt{g_{22}}}\frac{\partial \omega}{\partial \eta} + v\frac{\partial \omega}{\partial \zeta} + B_1 u^2 + B_2\omega^2 + B_3 u\omega = B_4 + \nu\frac{\partial^2 \omega}{\partial \zeta^2}$$

$$\frac{\partial}{\partial \xi}\left[\sqrt{\frac{g}{g_{11}}}\,u\right] + \frac{\partial}{\partial \eta}\left[\sqrt{\frac{g}{g_{22}}}\,\omega\right] + \sqrt{g}\frac{\partial v}{\partial \zeta} = 0$$

The boundary conditions for this system of equations are

$$u = v = w = 0 \quad \text{for} \quad \zeta = 0$$
$$u \to u_e, \quad w \to w_e \quad \text{for} \quad \zeta \to \infty \qquad (1.2)$$

Here ξ, η are coordinates on the surface of the body, $\zeta = 0$ is the equation of the surface; u, w and v are the velocity components along the ξ, η and ζ axes, respectively; p is the pressure; ν is the kinematic coefficient of viscosity; g_{11}, g_{22}, g_{12} are metric coefficients; $g = g_{11}g_{22} - g_{12}^2$. The coefficients A_1 through B_4 are determined by the external flow and the geometry of the body (see [2]).

We reduce the system of equations of the three-dimensional boundary layer to a form suitable for integration, and we introduce the following change of variables:

$$u = u_e(\xi, \eta)\, E(\xi, \eta, \lambda)$$
$$w = \beta(\xi, \eta)\, u_e(\xi, \eta)\,[G(\xi, \eta, \lambda) + \varphi E(\xi, \eta, \lambda)]$$

$$v = \sqrt{\frac{u_e(\xi,\eta)\nu}{\alpha(\xi,\eta)}}\left[K(\xi,\eta,\lambda) - \frac{\alpha}{\sqrt{g_{11}}}E\frac{\partial\lambda}{\partial\xi} - \frac{\alpha\beta}{\sqrt{g_{22}}}(G+\varphi E)\frac{\partial\lambda}{\partial\eta}\right]$$
$$\lambda = \sqrt{\frac{u_e}{\alpha\nu}}\zeta$$

where α and β are arbitrary functions, the choice of which is made below. Then the system of equations (1.1) can be reduced to the following form (see [2]):

$$\frac{\partial^2 E}{\partial\lambda^2} = K\frac{\partial E}{\partial\lambda} + N_1^*(E^2-1) + N_2^*G^2 + N_3^*EG + \tag{1.3}$$
$$N_4E\frac{\partial E}{\partial\xi} + N_5(G+\varphi E)\frac{\partial E}{\partial\eta}$$
$$\frac{\partial^2 G}{\partial\lambda^2} = K\frac{\partial G}{\partial\lambda} + M_1^*(E^2-1) + M_2^*G^2 + M_3^*EG +$$
$$N_4E\frac{\partial G}{\partial\xi} + N_5(G+\varphi E)\frac{\partial G}{\partial\eta}$$
$$-\frac{\partial K}{\partial\lambda} = P_1^*E + P_2^*G + N_4\frac{\partial E}{\partial\xi} + N_5\frac{\partial G}{\partial\eta} + \varphi N_5\frac{\partial E}{\partial\eta}$$

The boundary conditions assume the form

$$E = G = K = 0 \quad \text{for} \quad \lambda = 0 \tag{1.4}$$
$$E \to 1, \quad G \to 0 \quad \text{for} \quad \lambda \to \infty$$

The coefficients N_1^*, N_2^*, N_3^*, M_1^*, M_2^*, M_3^*, P_1^*, P_2^*, N_4, N_5 depend only on ξ and η ; they are associated with the geometry of the surface and the external flow. We integrate the transformed momentum equations of the system (1.3) with respect to the variable λ_1 from some value λ to ∞, taking the boundary conditions (1.4) into account. We find

$$-\frac{\partial E}{\partial\lambda} = -K(E-1) + (P_1^* + N_1^*)\theta_{11} + (P_2^* + N_3^*)\theta_{21} + \tag{1.5}$$
$$N_2^*\theta_{22} + N_1^*\theta_1 - P_2^*\theta_2 + N_4\frac{\partial\theta_{11}}{\partial\xi} + \varphi N_5\frac{\partial\theta_{11}}{\partial\eta} + N_5\frac{\partial\theta_{12}}{\partial\eta}$$
$$-\frac{\partial G}{\partial\lambda} = -KG + M_1^*(\theta_{11}+\theta_1) + (P_1^* + M_3^*)\theta_{21} +$$
$$(P_2^* + M_2^*)\theta_{22} + N_4\frac{\partial\theta_{21}}{\partial\xi} + N_5\frac{\partial\theta_{22}}{\partial\eta} + \varphi N_5\frac{\partial\theta_{21}}{\partial\eta}$$

Here we have used the asymptotic tendency to zero of the derivatives $\partial E/\partial\lambda$ and $\partial G/\partial\lambda$ for $\lambda \to \infty$. In addition, we assume existence of the integrals

$$\theta_{11} = \int_\lambda^\infty (E-1)E\,d\lambda_1, \quad \theta_{21} = \int_\lambda^\infty EG\,d\lambda_1, \quad \theta_{22} = \int_\lambda^\infty G^2\,d\lambda_1$$
$$\theta_{12} = \int_\lambda^\infty (E-1)G\,d\lambda_1, \quad \theta_1 = \int_\lambda^\infty (E-1)\,d\lambda_1, \quad \theta_2 = \int_\lambda^\infty G\,d\lambda_1$$

If the integration is made from zero to infinity, the equations pass over into the Kármán relations. On the left sides of Eqs. (1.5) we obtain components proportional to the frictional stress at the wall. Integrating Eqs. (1.5) a second time termwise across the boundary layer from zero to a current value λ and using the boundary conditions, for $\lambda = 0$, we obtain a system of nonlinear integro-differential equations

$$-E = \theta_{01}^* + (P_1^* + N_1^*)\,\theta_{11}^* + N_2^*\theta_{22}^* + (P_2^* + N_3^*)\,\theta_{21}^* + \tag{1.6}$$
$$N_1^*\theta_1^* - P_2^*\theta_2^* + N_4\frac{\partial}{\partial\xi}\theta_{11}^* + \varphi N_5\frac{\partial}{\partial\eta}\theta_{11}^* + N_5\frac{\partial\theta_{12}^*}{\partial\eta}$$
$$-G = \theta_{02}^* + M_1^*(\theta_{11}^* + \theta_1^*) + (P_2^* + M_2^*)\,\theta_{22}^* +$$
$$(P_1^* + M_3^*)\,\theta_{21}^* + N_4\frac{\partial}{\partial\xi}\theta_{21}^* + N_5\frac{\partial}{\partial\eta}\theta_{22}^* + \varphi N_5\frac{\partial}{\partial\eta}\theta_{21}^*$$

We have used the boundary conditions: $E = G = 0$ for $\lambda = 0$ and we have assumed the possibility of exchanging the order of integration and differentiation. Here

$$\theta_{01}^* = \int_0^\lambda K(1-E)\,d\lambda_1, \qquad \theta_{02}^* = -\int_0^\lambda KG\,d\lambda_1 \tag{1.7}$$

$$\theta_{11}^* = \int_0^\lambda\int_{\lambda_1}^\infty (E-1)\,Ed\lambda_2\,d\lambda_1, \quad \theta_{22}^* = \int_0^\lambda\int_{\lambda_1}^\infty G^2d\lambda_2\,d\lambda_1$$

$$\theta_{21}^* = \int_0^\lambda\int_{\lambda_1}^\infty EGd\lambda_2\,d\lambda_1 \qquad \theta_2^* = \int_0^\lambda\int_{\lambda_1}^\infty Gd\lambda_2\,d\lambda_1$$

$$\theta_{12}^* = \int_0^\lambda\int_{\lambda_1}^\infty (E-1)\,Gd\lambda_2\,d\lambda_1, \quad \theta_1^* = \int_0^\lambda\int_{\lambda_1}^\infty (E-1)\,d\lambda_2\,d\lambda_1$$

A change in the magnitudes of the velocity depends on terms appearing in the expression (1.6) which characterize the influence of factors of a diverse nature. This variation is associated with the nonlinear interaction of the longitudinal flow, the transverse flow, and the flow perpendicular to the wall, and also the body geometry and the external pressure.

The qualitiative nature of the influence of these factors on the value of the magnitude of the velocity is a varied one. The main contribution to the profile of the longitudinal velocitv component is found to consist of the same terms as for a two-dimensional flow (θ_{01}^*, $(P_1^* + N_1^*)\,\theta_{11}$, $N_1^*\theta_1^*$). The term $P_2^*\theta_2^*$ characterizes the fundamental influence of the secondary flow on the magnitude of the longitudinal velocity. The term $(P_2^* + N_3^*)\,\theta_{21}^*$ in the first of the Eqs. (1.6) is the result of the interaction of the flows in the longitudinal and transverse directions. The term $N_2^*\theta_{22}^*$ indicates the nonlinear influence of the secondary flow on the velocity profile.

The main contribution to the magnitude of the transverse component of the velocity arises from the terms θ_{02}^*, $M_1^*(\theta_{11}^* + \theta_1^*)$. If the quantity M_1^* is equal to zero, then the secondary flow disappears. If the quantity M_1^* is comparatively small, then the secondary flow is also small. The terms $(P_1^* + M_3^*)\,\theta_{21}^*$, $(P_2^* + M_2^*)\,\theta_{22}^*$ characterize the interaction of the flows in the longitudinal and transverse directions. The results of finite-difference calculations of boundary layer flows, obtained earlier (see [2]) for a number of problems (ellipsoids at an angle of attack, direct and inverted cones at an angle of attack, and other bodies), show that the quantities θ_{01}^*, θ_{02}^*, θ_{11}^*, θ_{22}^*, θ_{21}^*, θ_{12}^*, θ_1^*, θ_2^* and the quantities θ_{11}, θ_{22}, θ_{21}, θ_{12}, θ_1, θ_2 vary comparatively weakly along the surface of the body, although there is a fairly noticeable variation of the velocity profile. In addition, the integrals θ_{22}^*, θ_{21}^*, θ_{12}^*, θ_2^* and the integrals θ_{22}, θ_{21}, θ_{12}, θ_2 are considerably less than the integrals $\theta_{11}{}^*$, $\theta_1{}^*$ and θ_{11}, θ_1 in the region of the "nonseparable" boundary layer.

2. To solve the system of nonlinear integro-differential equations (1.6) we apply the method of successive approximations [3]. In what follows, we consider the problem in the locally self-similar case, based on the assumption that the derivatives of the functions E and G along the ξ and η coordinates are small when self-similar variables are used and that these derivatives can be neglected in Eqs. (1.6), i.e. the coordinates ξ and η enter the solution of the problem as parameters depending on the external flow and the geometry of the body.

The boundary condition for $\lambda = 0$, as a consequence of the Eqs. (1.6), is satisfied automatically. In order for the boundary conditions at the outer side of the boundary layer ($\lambda \to \infty$) to be satisfied in the successive approximations process, we introduce unknown "governing" functions $c^{(n)}(\xi,\ \eta)$ and $b^{(n)}(\xi,\ \eta)$ as follows:

$$E^{(n)} = E(\xi,\ \eta,\ c^{(n)}(\xi,\ \eta)\lambda),$$
$$G^{(n)} = b^{(n)}(\xi,\ \eta)\, G(\xi,\ \eta,\ c^{(n)}(\xi,\ \eta)\lambda)$$

For the approach proposed we obtain, at each nth step of the iterational process, for $\lambda \to \infty$ equations for $c^{(n)}$ and $b^{(n)}$. If the iterational process converges for $n \to \infty$, the quantities $c^{(n)}$ and $b^{(n)}$ tend to unity.

In the locally self-similar case the successive approximations algorithm has the form ($\sqrt{\delta^{(n)}} = 1 / c^{(n)}$):

$$-E_a^{(n+1)} = \delta^{(n)}(A_{1a}^{(n)} + b^{(n)}B_{1a}^{(n)} + b^{(n)2}C_{1a}^{(n)}) \tag{2.1}$$
$$-G_a^{(n+1)} = \delta^{(n)}(A_{2a}^{(n)} + b^{(n)}B_{2a}^{(n)} + b^{(n)2}C_{2a}^{(n)})$$

Here

$$A_{1a}^{(n)} = -P_1{}^*\int_0^\zeta (1 - E^{(n)}) f_1 d\zeta + (P_1{}^* + N_1{}^*)\,\theta_{11}^{*(n)} + N_1{}^*\theta_1^{*(n)} \tag{2.2}$$

$$B_{1a}^{(n)} = -P_2{}^*\int_0^\zeta (1 - E^{(n)}) f_2 d\zeta + (P_2{}^* + N_3{}^*)\,\theta_{21}^{*(n)} - P_2{}^*\theta_2^{*(n)}$$

$$C_{1a}^{(n)} = N_2{}^*\theta_{22}^{*(n)}, \qquad A_{2a}^{(n)} = M_1{}^*(\theta_{11}^{*(n)} + \theta_1^{*(n)})$$

$$B_{2a}^{(n)} = P_1{}^*\int_0^\zeta G^{(n)} f_1 d\zeta + (P_1{}^* + M_3{}^*)\,\theta_{21}^{*(n)}$$

$$C_{2a}^{(n)} = P_2{}^*\int_0^\zeta G^{(n)} f_2 d\zeta + (P_2{}^* + M_2{}^*)\,\theta_{22}^{*(n)}$$

$$f_1 = \int_0^\zeta E d\zeta, \qquad f_2 = \int_0^\zeta G d\zeta \qquad (\zeta = c^{(n)}\lambda)$$

The dimensionless friction components at the wall are obtained from the formulas

$$-\frac{\partial E^{(n+1)}}{\partial\lambda}\bigg|_{\lambda=0} = \sqrt{\delta^{(n)}}\{[N_1{}^*(\theta_{11.0}^{(n)} + \theta_{1.0}^{(n)}) + P_1{}^*\theta_{11.0}^{(n)}] + \tag{2.3}$$
$$b^{(n)}[(P_2{}^* + N_3{}^*)\,\theta_{21.0}^{(n)} - P_2{}^*\theta_{2.0}^{(n)}] + b^{(n)2}N_2{}^*\theta_{22.0}^{(n)}\}$$

$$-\frac{\partial G^{(n+1)}}{\partial \lambda}\bigg|_{\lambda=0} = \sqrt{\delta^{(n)}}\{M_1^*(\theta_{11.0}^{(n)} + \theta_{1.0}^{(n)}) + b^{(n)}(P_1^* + M_3^*)\theta_{21.0}^{(n)} + b^{(n)2}(P_2^* + M_2^*)\theta_{22.0}^{(n)}\}$$

The values of the "governing" functions $b^{(n)}$ and $\delta^{(n)}$ are obtained from the expressions

$$\delta^{(n)} = (A_{1a\infty}^{(n)} + b^{(n)}B_{1a\infty}^{(n)} + b^{(n)2}C_{1a\infty}^{(n)})^{-1} \tag{2.4}$$

$$b^{(n)2}C_{2a\infty}^{(n)} + b^{(n)}B_{2a\infty}^{(n)} + A_{2a\infty}^{(n)} = 0$$

From the second of Eqs. (2.4) we find

$$b^{(n)} = (-B_{2a\infty}^{(n)} \pm (B_{2a\infty}^{(n)2} - 4A_{2a\infty}^{(n)}C_{2a\infty}^{(n)})^{1/2}) / 2C_{2a\infty}^{(n)} \tag{2.5}$$

The sign in the expression for $b^{(n)}$ is chosen in such a way that in the axially symmetric case $b^{(n)} \equiv 0$, i.e. the plus sign is chosen since in the axially symmetric case $A_{2a\infty} = 0$. The condition for the existence of secondary flows leads to relationships of the form

$$B_{2a\infty}^{(n)2} - 4A_{2a\infty}^{(n)}C_{2a\infty}^{(n)} \geqslant 0 \tag{2.6}$$

The coefficients $A_{1a\infty}^{(n)}$, $B_{1a\infty}^{(n)}$, $C_{1a\infty}^{(n)}$ and $A_{2a\infty}^{(n)}$, $B_{2a\infty}^{(n)}$, $C_{2a\infty}^{(n)}$ are functions of ξ and η. The relations (2.6) connect the coefficients appearing in the initial system of equations and are determined by the nature of the external flow and the body geometry. From the relations (2.6) we obtain a connection between the parameters of the external flow and the body geometry for which a solution can be constructed by the method indicated.

The successive approximations algorithm adduced can be implemented in various ways, in particular, with the help of an electronic computer. Since the difficulties of numerically implementing this algorithm are of the same, and possibly even greater, order as those of the finite-difference method, the fundamental aim of our approach is to obtain a solution in analytical form. With this aim in view, we consider the first approximation. An analytical version of the successive approximations method for solving such involved problems, is justified providing it yields a solution close to the desired solution even in the first approximation.

3. We shall define the zeroth approximation in the class of the following functions $\{Z_m\}$ (see [4]):

$$Z_m(y) = \frac{A_m}{m!}\int_\infty^y (y-\xi)^m e^{-\xi^2}\, d\xi, \qquad m = 0, 1, 2, \ldots \tag{3.1}$$

The coefficients A_m are chosen so that $Z_m(0) = 1$. Then

$$A_{-1} = 1, \quad A_0 = -\frac{2}{\sqrt{\pi}}, \quad A_1 = 2, \quad A_{2k} = -\frac{2^{k+1}}{\sqrt{\pi}}(2k)!! \tag{3.2}$$

$$A_{2k-1} = (2k-1)!!\, 2^k, \qquad A_k = 2kA_{k-2}$$

This system of functions satisfies the relations

$$Z_m = Z_{m-2} + \frac{A_m}{mA_{m-1}^*}yZ_{m-1} \quad (Z_{-1}(y) = e^{-y^2}), \qquad \frac{d^kZ_m}{dy^k} = \frac{A_m}{A_{m-k}}Z_{m-k} \tag{3.3}$$

$$\int_\infty^y Z_m(y)\,dy = \frac{A_m}{A_{m+1}}Z_{m+1}, \qquad \int_0^y Z_m(y)\,dy = \frac{A_m}{A_{m+1}}(Z_{m+1} - 1)$$

We note that the system of functions $\{Z_m\}$ also possesses properties which make it possible to calculate integrals of the form

$$I_{p,q} = \int_0^y Z_p(y) Z_q(y)\, dy \tag{3.4}$$

in terms of the initial system of functions. We have

$$I_{p,q} = -\frac{A_p}{A_{q+1}}(1 - Z_p Z_{q+1}) - \frac{A_p A_q}{A_{p-1} A_{q+1}} I_{p-1,\,q+1}$$

Moreover,

$$\int_0^y Z_m Z_{m-1}\, dy = \frac{A_{m-1}}{2A_m}(Z_m{}^2 - 1) \tag{3.5}$$

We define the zeroth approximation for the functions $E^{(0)}$ and $G^{(0)}$ in the form

$$E^{(0)} = 1 - Z_0(\zeta), \qquad G^{(0)} = b^{(0)}[Z_0(\zeta) - Z_{-1}(\zeta)], \qquad \zeta = \lambda / \sqrt{\delta^{(0)}} \tag{3.6}$$

Then in the locally self-similar case the first approximation is as follows:

$$\begin{aligned} -E_a^{(1)} &= \delta^{(0)}(A_{1a}^{(0)} + b^{(0)} B_{1a}^{(0)} + b^{(0)\,2} C_{1a}^{(1)}) \\ -G_a^{(1)} &= \delta^{(0)}(A_{2a}^{(0)} + b^{(0)} B_{2a}^{(0)} + b^{(0)\,2} C_{2a}^{(0)}) \end{aligned} \tag{3.7}$$

where $\delta^{(0)}$ and $b^{(0)}$ are determined from the expressions

$$\begin{aligned} \delta^{(0)} &= -(A_{1a\infty}^{(0)} + b^{(0)} B_{1a\infty}^{(0)} + b^{(0)2} C_{1a\infty}^{(0)})^{-1} \\ b^{(0)} &= [-B_{2a\infty}^{(0)} + (B_{2a\infty}^{(0)2} - 4A_{2a\infty}^{(0)} \cdot C_{2a\infty}^{(0)})^{1/2}]/2C_{2a\infty}^{(0)} \end{aligned} \tag{3.8}$$

Here

$$\begin{aligned} A_{1a}^{(0)} = &-P_1{}^* \frac{A_0}{A_1}\left[\frac{A_1}{A_2}(Z_2 - 1) - \frac{A_{-1}}{A_0}(Z_0 - 1) - I_{1,0} + \frac{A_0}{A_1}(Z_1 - 1)\right] + \\ &(P_1{}^* + N_1{}^*)\left(J_{0,0} + \frac{A_0}{A_2}(Z_2 - 1)\right) + N_1{}^* \frac{A_0}{A_2}(Z_2 - 1) \end{aligned} \tag{3.9}$$

$$\begin{aligned} B_{1a}^{(0)} = \;&P_2{}^*\left(\frac{A_{-1}}{A_0} I_{0,0} - \frac{A_0}{A_1} I_{1,0} + \left(\frac{A_0}{A_1} - \frac{A_{-1}}{A_0}\right)\frac{A_0}{A_1}(Z_1 - 1)\right) + \\ &(P_2{}^* + N_3{}^*)\left(\frac{A_{-1}}{A_1}(Z_1 - 1) - \frac{A_0}{A_2}(Z_2 - 1) - J_{0,0} + J_{0,-1}\right) - \\ &P_2{}^*\left(\frac{A_{-1}}{A_1}(Z_1 - 1) - \frac{A_0}{A_2}(Z_2 - 1)\right) \end{aligned}$$

$$C_{1a}^{(0)} = N_2{}^*(J_{0,0} - 2J_{0,-1} + J_{-1,-1})$$

$$\begin{aligned} J_{0,0} = &\frac{A_0{}^2}{2A_1{}^2}(1 - Z_1{}^2) - \left(\frac{2}{\sqrt{\pi}} + \frac{A_0}{A_1}\right)\frac{A_{-1}}{2A_0}(1 - Z_0{}^2) - \\ &\frac{1}{\sqrt{\pi}}\frac{A_0}{A_1}(Z_1(\sqrt{2}) - 1) \end{aligned}$$

$$J_{0,-1} = -\frac{A_{-1}}{2A_0} I_{0,0}, \qquad J_{-1,-1} = \frac{\sqrt{\pi}}{4}\frac{A_0}{A_1}(Z_1(\sqrt{2}) - 1)$$

(The notation $Z_1(\sqrt{2})$ indicates that the argument is multiplied by the quantity $\sqrt{2}$.)

Analogously, we obtain the quantities $A_{2a}^{(0)}$, $B_{2a}^{(0)}$, and $C_{2a}^{(0)}$

$$A_{2a}^{(0)} = M_1^* \left(J_{0,0} + 2\frac{A_0}{A_2}(Z_2 - 1)\right) \tag{3.10}$$

$$B_{2a}^{(0)} = P_1^* \left(\frac{A_0}{A_2}(Z_2 - 1) - 2\frac{A_{-1}}{A_1}(Z_0 - 1) + \frac{1}{2}(Z_{-1} - 1) + \left(\frac{A_0}{A_1}\right)^2 (Z_1 - 1) - \frac{A_0}{A_1}(I_{1,0} - I_{1,-1})\right) + (P_1^* + M_3^*) \times \left(\frac{A_{-1}}{A_1}(Z_1 - 1) - \frac{A_0}{A_2}(Z_2 - 1) - J_{0,0} + J_{0,-1}\right)$$

$$C_{2a}^{(0)} = P_2^* \left\{\left[I_{0,-1}\frac{A_{-1}}{A_0} - I_{0,0}\frac{A_{-1}}{A_0} + I_{1,0}\frac{A_{-1}}{A_1} - I_{1,-1}\frac{A_0}{A_1}\right] - \left[\frac{A_0}{A_1} - \frac{A_{-1}}{A_0}\right]\left[\frac{A_0}{A_1}(Z_1 - 1) - \frac{A_{-1}}{A_0}(Z_0 - 1)\right]\right\} + (P_2^* + M_2^*)(J_{0,0} - 2J_{0,-1} + J_{-1,-1})$$

Thus we have obtained, in a first approximation, the solution in the general case for an arbitrary external flow and arbitrary body geometry.

We find now the quantities $A_{1a\infty}^{(0)}$, $B_{1a\infty}^{(0)}$, $C_{1a\infty}^{(0)}$ and $A_{2a\infty}^{(0)}$, $B_{2a\infty}^{(0)}$, $C_{2a\infty}^{(0)}$. We have

$$-A_{1a\,\infty}^{(0)} = \frac{1}{4}P_1^* + \left(\frac{1}{2\pi} + \frac{1}{4}\right)N_1^*, \tag{3.11}$$

$$-B_{1a\,\infty}^{(0)} = \left(\frac{1}{4} - \frac{\sqrt{2}}{4}\right)P_2^* + \left(\frac{\sqrt{2}}{4} - \frac{1}{2\pi}\right)N_3^*$$

$$-C_{1a\,\infty}^{(0)} = -N_2^*\left[\frac{\sqrt{2}-1}{2} - \frac{1}{2\pi}\right]$$

In an analogous way we obtain the quantities $A_{2a\infty}^{(0)}$ $B_{2a\infty}^{(0)}$ and $C_{2a\infty}^{(0)}$

$$-A_{2a\,\infty}^{(0)} = \left(\frac{1}{2\pi} + \frac{1}{4}\right)M_1^* \tag{3.12}$$

$$-B_{2a\,\infty}^{(0)} = -3\left(\frac{1}{4} - \frac{\sqrt{2}}{4}\right)P_1^* + \left(\frac{\sqrt{2}}{4} - \frac{1}{2\pi}\right)M_3^*$$

$$-C_{2a\,\infty}^{(0)} = -P_2^*\left(\frac{\sqrt{2}}{2} + \frac{\pi}{8} - 1\right) - M_2^*\left(\frac{\sqrt{2}-1}{2} - \frac{1}{2\pi}\right)$$

In final form the formulas for the dimensionless friction components at the wall are written in the form

$$\left.\frac{\partial E^{(1)}}{\partial\lambda}\right|_{\lambda=0} = \sqrt{\delta^{(0)}}\{0.2337P_1^* + 0.7978N_1^* + b^{(0)}(0.2095N_3^* - 0.1125P_2^*) - 0.071b^{(0)2}N_2^*\} \tag{3.13}$$

$$\left.\frac{\partial G^{(1)}}{\partial\lambda}\right|_{\lambda=0} = \sqrt{\delta^{(0)}}\{0.7978M_1^* + b^{(0)}0.2095(P_1^* + M_3^*) - 0.071b^{(0)2}(P_2^* + M_2^*)\}$$

where the quantities $\delta^{(0)}$ and $b^{(0)}$ are computed from the formulas

$$b^{(0)} = [0.31P_1^* + 0.194M_3^* - ((0.31P_1^* + 0.194M_3^*)^2 + 1.636M_1^*(0.1P_2^* + 0.047M_2^*))^{1/2}]/(0.199P_2^* + 0.095M_2^*) \tag{3.14}$$

$$\delta^{(0)} = (0.25P_1{}^* + 0.409N_1{}^* + b^{(0)}(0.194N_3{}^* - 0.103P_2{}^*) - b^{(0)2}0.048N_2{}^*)^{-1}$$

Components of the friction magnitudes at the wall are determined from the formulas ($g_{12} = 0$)

$$\tau_1 = \mu\left(\frac{\partial u}{\partial \zeta}\right)_{\zeta=0} = \mu\sqrt{\frac{u_e^3}{\nu\alpha}}\left(\frac{\partial E}{\partial \lambda}\right)_{\lambda=0}$$

$$\tau_2 = \mu\left(\frac{\partial \omega}{\partial \zeta}\right)_{\zeta=0} = \mu\beta\sqrt{\frac{u_e^3}{\nu\alpha}}\left(\left(\frac{\partial G}{\partial \lambda}\right)_{\lambda=0} + \varphi\left(\frac{\partial E}{\partial \lambda}\right)_{\lambda=0}\right)$$

$$(\alpha = \xi, \ \beta = \eta)$$

4. We consider the flow around the stagnation point of a three-dimensional body with two-fold curvature. Let M be a stagnation point on the surface of a smooth body. We assume that the external flow is irrotational. About the point M the surface of the body may be represented by its tangent plane. We choose a rectangular system of coordinates (ξ, η, ζ) so that the ξ and η axes lie in this plane and the ζ-axis is perpendicular to it. The velocity components of the external flow have the form

$$u_e = a\xi, \qquad w_e = b\eta$$

Following [2], we choose $\beta = \eta / \xi$, $\alpha = \xi$. Then $\varphi = b / a$. If $\varphi = 0$, this case corresponds to two-dimensional flows; if $\varphi = 1$, it corresponds to a flow around a body of revolution, symmetrically placed in the flow. We consider the case in which $-1 < \varphi \leqslant 1$.

The dimensionless friction magnitude at the stagnation point is obtained from the formulas (3.13) and, in the given case, assumes the form

$$\frac{\partial E^{(1)}}{\partial \lambda} = \sqrt{\delta^{(0)}}(1.0315 + 0.2337\varphi - 0.1125b^{(0)}) \tag{4.1}$$

$$\frac{\partial G^{(1)}}{\partial \lambda} = \sqrt{\delta^{(0)}}(-0.7978\varphi(1-\varphi) + 0.2095(1+3\varphi)b^{(0)} - 0.142b^{(0)2})$$

Here the quantities $\delta^{(0)}$ and $b^{(0)}$ are computed from the formulas

$$\delta^{(0)} = (0.659 + 0.25\varphi - 0.103b^{(0)})^{-1} \tag{4.2}$$

$$b^{(0)} = (0.31 + 0.698\varphi \pm ((0.31 + 0.698\varphi)^2 + (\varphi^2 - \varphi)0.24)^{1/2})/0.295$$

We observe that two values are obtained for $b^{(0)}$. The quantity under the radical sign is strictly positive for $-1 < \varphi \leqslant 1$, i.e. for each value in the given interval there exist two distinct solutions satisfying all the required conditions. One solution, which has a physical meaning, is easily selected from the condition that secondary flows are not present in the axially symmetric case ($b^{(0)} \equiv 0$). The other solution is obtained by choosing the plus sign in front of the radical. This solution is probably of mathematical interest. In performing the numerical calculations, the existence of a second solution may lead to a situation where, specifying an initial profile, we may find the solution at the stagnation point incorrectly. In the iterative process of numerically solving the problem it is possible to obtain two distinct solutions. But as soon as the prifile of the axially symmetric problem is taken as the initial profile, the iterative process converges rapidly to the required solution. We give below a comparison of the results from [1], obtained by numerically integrating a system analogous to the system of Eqs. (1.3) with boundary conditions (1.4), with the results obtained from the first approximation.

φ	0.00	0.25	0.50	0.75	1.00
$E'(0)$	1.233	1.247	1.267	1.288	1.312
$E_a'(0)$	1.270	1.276	1.290	1.307	1.327
$G'(0)$	0.570	0.805	0.998	1.164	1.312
$G_a'(0)$	0.610	0.832	1.017	1.180	1.327

The results indicated with the subscript a were obtained from the approximate analytical formulas (4.1) and (4.2) of the present paper. The good agreement of the friction components, even in the first approximation, throughout the range of variation of the parameter φ from zero to unity should be noted.

We restrict ourselves to this range of variation of φ, since for other values ($\varphi < 0$ and $\varphi > 1$) we have the following relationships ($E = f'$, $G = g'$):

$$f(\lambda, -\varphi) = f(\lambda, \varphi), \qquad f'(\lambda, -\varphi) = f'(\lambda, \varphi)$$
$$g(\lambda, -\varphi) = -g(\lambda, \varphi), \qquad g'(\lambda, -\varphi) = g'(\lambda, \varphi)$$

and

$$f(\lambda, 1/\varphi) = f^{1/2}(\lambda/\varphi^{1/2}, \varphi), \quad f'(\lambda, 1/\varphi) = -g'(\lambda/\varphi^{1/2}, \varphi)$$
$$g(\lambda, 1/\varphi) = f^{1/2}, (\lambda/\varphi^{1/2} \varphi), \quad g'(\lambda, 1/\varphi) = f'(\lambda/\varphi^{1/2}, \varphi) \; (\varphi > 0)$$

Using the values of the functions $\{Z_m\}$, we can construct the velocity profiles in the longitudinal and transverse directions and also in the direction normal to the wall.

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